A MAXIMUM PRINCIPLE FOR CONTROLLED STOCHASTIC FACTOR MODEL

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Abstract. In the present work, we consider an optimal control for a three-factor stochastic factor model. We assume that one of the factors is not observed and use classical filtering technique to transform the partial observation control problem for stochastic differential equation (SDE) to a full observation control problem for stochastic partial differential equation (SPDE). We then give a sufficient maximum principle for a system of controlled SDEs and degenerate SPDE. We also derive an equivalent stochastic maximum principle. We apply the obtained results to study a pricing and hedging problem of a commodity derivative at a given location, when the convenience yield is not observable.

1. Introduction

The use of stochastic factor model in stock price modeling has increased in the recent years in the financial mathematics' literature (see for example [4, 7, 9] and references therein). This is due to the fact that the dynamics of the underlying commodity (stock) could depend on a stochastic external economic factor which may or may not be traded directly. Let us for example consider the hedging problem of a commodity derivative at a given location that faces an agent, when the convenience yield is not observed; see for example [4]. It may happen that there is no market in which the commodity can be traded directly. Hence the agent needs to trade similar asset and thus faces the basis risk which may depend on factors such as market demand, transportation cost, storage cost, etc. The presence of the risk associated to the location and which cannot be perfectly hedge makes the market incomplete. In this situation, it is not always possible to have an exact replication of the derivative. One way to overcome this difficulty is through utility indifference pricing. The method consists of finding the initial price p of a claim Π that makes the buyer of the contract utility indifferent, that is, buying

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the contract with initial price p and with the right to receive the claim Π at maturity or not buying the contract and receive nothing. Due to the unobserved factor, the above optimisation problems can be seen as problems of optimal control for partially observed systems. There are three existing methods to solve such problem in the literature: the duality approach, the dynamic programming and the maximum principle; see e.g., [1, 2, 3, 9, 15, 19, 20, 23, 25, 26] and references therein. When using dynamic programming, the value function satisfies a non-linear partial differential equation known as the Hamilton-Jacobi-Bellman which does not always admits a classical solution. Moreover, it does not give necessary condition for optimality unless the value function is continuously differentiable.

In this paper, we use the stochastic maximum principle to solve an optimal control problem for the given stochastic factor model when the factor is not observable. The factor is replaced by its conditional distribution and we use filtering theory to transform the partial observation control problem for (ordinary) stochastic differential equation to a full observation control problem for stochastic partial differential equation (for more details on filtering theory see for example [1, 2]). Since the state (or signal process) and the observation process are correlated, the diffusion operator in the derived unnormalized density depends on its first order derivatives. This leads to a degenerate controlled stochastic partial differential equation and the sufficient stochastic maximum principle obtained in [22, 23] cannot directly be applied in this paper. Tang in [25] also studies a problem of partially observed systems using stochastic maximum principle. However, he uses Bayes' formula and Girsanov theorem to obtain a related control problem while here we use an approach based on Zakai's equation of the unnormalized density. In addition, the value function in [25] only depends on the signal process. Our setting also covers that of [22] since we have a more general controlled stochastic partial differential equation for the system in full information. Our setting is related to [26], where the author derives a "weak" necessary maximum principle for an optimal control problem for stochastic partial differential equations. The author shows existence and uniqueness of generalised solution of the controlled process and the associated adjoint equation. In the same direction, let us also mention the interesting book [17], where the authors solve a "strong" necessary maximum principle for evolution equations in infinite dimension. The operator is assumed to be unbounded and in contrary to [26], the diffusion coefficient does not depend on the first order derivative of the state process. Our result can be seen as a "strong" sufficient stochastic maximum principle, since we assume existence of strong solution of the associated degenerate controlled stochastic partial differential equation. Conditions on existence and uniqueness of strong solutions for such SPDE can be found in [8]. In fact, assuming some regularity on the coefficients of the controlled processes, the profit rate and the bequest functions of the performance functional, there exists a unique strong classical solution for the backward stochastic partial differential equation representing the associated adjoint processes; see e.g., [5] and references therein. Note that the particular setup identified by [26] (or [17]) can be derived from our setup as well and in this case, the resulting Hamiltonians are the same, and so are their associated adjoint processes. The sufficient maximum principle obtained in this work is used to solve a problem of utility maximization for stochastic factor model.

The sufficient maximum principle presented in this paper requires some concavity assumptions which may not be satisfied in some applications. To overcome this situation, we also present an equivalent maximum principle for degenerate stochastic partial differential equation which does not require concavity assumption.

The paper is organised as follows: In Section 2, we motivate and formulate the control problem. In Section 3, we derive a sufficient and an equivalent stochastic maximum principle for degenerate stochastic partial differential equation. In Section 4, we apply the obtained results to solve a hedging and pricing problem for a commodity derivative at a given location when the convenience yield is not observable.

2. Model and problem formulation

2.1. A motivative example. In this section, we motivate the problem by briefly summarizing the classical Gibson-Schwartz two-factor model for commodity and convenience yield (see for example [7] and [4] for unobservable yield). Let us fix a time interval horizon $[0,T]$. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be a complete filtered probability space on which are given two correlated standard Brownian motions $W^1(t) = \{W^1(t), t \in [0,T]\}$ and $W^2(t) = \{W^2(t), t \in [0,T]\}$ with correlation coefficient $\rho \in [-1,1]$.

We consider the replicating and pricing problem of an agent in a certain location who wishes to buy a contingent claim written on a commodity and that pays off $\Pi(S_*)$ at time T. Here S_* denotes the commodity spot price. Unfortunately there is no market for derivatives written on S_* and there can only be bought over-the-counter. One way is then to price and hedge the claim on a similar traded asset. However, using the corresponding traded asset exposes the agent to the basis risk, which can be seen as a function of several variables such as transportation cost, market demand, etc. One can think of the basis risk as a non traded location factor. Therefore, the claim depends on the commodity (traded asset) price $\tilde{S}$ and the non-traded location factor B, that is $\Pi = \Pi\left(\tilde{S}(T), B\right)$.

We assume that the dynamics of the convenience unobserved yield $Z(t) = \{Z(t), t \in [0,T]\}$ and the observed spot price $\tilde{S}(t) = \{\tilde{S}(t), t \in [0,T]\}$ are respectively given by the following stochastic differential equations (SDEs for short)

$$\mathrm{d}\tilde{S}(t) = (r(t) - Z(t))\,\tilde{S}(t)\mathrm{d}t + \sigma\tilde{S}(t)\mathrm{d}W^1(t) \tag{2.1}$$

and

$$\mathrm{d}Z(t) = k\,(\theta - Z(t))\,\mathrm{d}t + \gamma\mathrm{d}W^2(t). \tag{2.2}$$

From now on, we will often use $Y(t) = \log\tilde{S}(t)$, then (2.1) and (2.2) become respectively

$$\mathrm{d}Y(t) = \left(r(t) - \frac{1}{2}\sigma^2 - Z(t)\right)\mathrm{d}t + \sigma\mathrm{d}W^1(t), \tag{2.3}$$

$$\mathrm{d}Z(t) = k\,(\theta - Z(t))\,\mathrm{d}t + \rho\gamma\mathrm{d}W^1(t) + \sqrt{1-\rho^2}\gamma\mathrm{d}W^{\perp}(t), \tag{2.4}$$

where $W^{\perp}(t) = \{W^{\perp}(t), t \in [0,T]\}$ is a standard Brownian motion on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ independent of $W^1(t)$. Let $r(t) = \{r(t), t \in [0,T]\}$ denote the short rate and assume that it is deterministic. Then the price of the riskless asset $S^0(t) = \{S^0(t), t \in [0,T]\}$ satisfies the following ordinary differential equation

$$\mathrm{d}S^0(t) = S^0(t)r(t)\mathrm{d}t. \tag{2.5}$$

Denote by $u(t) = \{u(t), t \in [0,T]\}$ the amount of wealth invested in the risky asset. We assume that $u(t)$ takes values is a given closed set $U \subset \mathbb{R}$. It follows from the self-financing condition that the dynamics of the wealth $X(t) = \{X(t), t \in [0,T]\}$ evolves according to the following SDE

$$\mathrm{d}X(t) = u(t)\frac{\mathrm{d}\tilde{S}(t)}{\tilde{S}(t)} + (1-u(t))\frac{\mathrm{d}S^0(t)}{S^0(t)},$$

that is

$$\mathrm{d}X(t) = (r(t)X(t) - Z(t)u(t))\,\mathrm{d}t + \sigma u(t)\mathrm{d}W^1(t), \;\; X(0) = x. \tag{2.6}$$

Using (2.3), the above equation becomes

$$\mathrm{d}X(t) = \Big(r(t)X(t) - (r - \frac{1}{2}\sigma^2)u(t)\Big)\mathrm{d}t + \sigma u(t)\mathrm{d}Y(t). \tag{2.7}$$

Recall that in this market, we are interested on a replicating and pricing problem of an economic agent who wishes to buy a contingent claim that pays off $\Pi(T)$ at time $T > 0$ in a given geographical location. The dependence of the claim Π on the location factor B makes the market incomplete and therefore perfect hedging is not possible. In this situation, the optimal portfolio can be chosen as the maximiser of the expected utility of the terminal wealth of the agent and the initial price of the claim can be derived via utility indifference pricing. The utility indifference price is given as follows: fix a utility function $U : \mathbb{R} \to (-\infty, \infty)$. The agent with initial wealth x and no endowment of the claim will simply face the problem of maximizing her expected utility of the terminal wealth $X^{x,u}(T)$; that is

$$V_0(x) = \sup_{u\in\mathcal{U}_{ad}} \mathbb{E}\Big[U\Big(X^{x,u}(T)\Big)\Big] = \mathbb{E}\Big[U\Big(X^{x,\hat{u}}(T)\Big)\Big], \tag{2.8}$$

where $\hat{u}$ is an optimal control (if it exists) and $\mathcal{U}_{ad}$ is the set of admissible controls to be defined later. The agent with initial wealth x and who is willing to pay p^b today for a unit of claim Π at time T faces the following expected utility maximization problem

$$\begin{aligned} V_\Pi(x - p^b) &= \sup_{u\in\mathcal{U}_{ad}} \mathbb{E}\Big[U\Big(X^{x-p,u}(T) + \Pi\Big(\tilde{S}(T), B\Big)\Big)\Big] \\ &= \mathbb{E}\Big[U\Big(X^{x-p,\hat{u}}(T) + \Pi\Big(\tilde{S}(T), B\Big)\Big)\Big]. \end{aligned} \tag{2.9}$$

The utility indifference pricing principle says that the *fair price* of the claim with payoff $\Pi\Big(\tilde{S}(T), B\Big)$ at time T is the solution to the equation

$$V_\Pi(x - p^b) = V_0(x). \tag{2.10}$$

We assume in this paper that the claim is a concave function. Example of such claims are forward contracts. Let $\mathcal{F}_t^{\tilde{S}} = \sigma\Big(\tilde{S}(t_1), 0 \leq t_1 \leq t\Big)$ be the σ-algebra generated by the commodity price, the set of admissible controls is given by

$$\begin{aligned} \mathcal{U}_{ad} =&\{u(t) : u \text{ is } \mathbb{F}^{\tilde{S}}\text{-progressively measurable };E[\int_0^T u^2(t)\mathrm{d}t] < \infty, \\ &X^{x,u}(t) \geq 0, \ \mathbb{P}\text{-a.s. for all } \ t \in [0, T]\}. \end{aligned} \tag{2.11}$$

Assumption A1. *The basis $B = B\Big(Z(T)\Big) + \bar{B}$, where B is a smooth function and $\bar{B}$ is a random variable independent of $\mathcal{F}_T$.*

Since $\bar{B}$ is independent of $\mathcal{F}_T$, we can rewrite (2.9) as follows:

$$\begin{aligned} V_\Pi(x) &= \sup_{u\in\mathcal{U}_{ad}} \mathbb{E}\Big[\int_{\mathbb{R}} U\Big(X^{u,x}(T) + \Pi\Big(\tilde{S}(T), B\Big(Z(T)\Big) + \bar{b}\Big)\Big)\mathrm{d}\mathbb{P}_{\bar{B}}\Big] \\ &= \mathbb{E}\Big[\int_{\mathbb{R}} U\Big(X^{\hat{u},x}(T) + \Pi\Big(\tilde{S}(T), B\Big(Z(T)\Big) + \bar{b}\Big)\Big)\mathrm{d}\mathbb{P}_{\bar{B}}\Big], \end{aligned} \tag{2.12}$$

where

$$\begin{cases} \mathrm{d}\ln\tilde{S}(t) = \Big(r(t) - \dfrac{1}{2}\sigma^2 - Z(t)\Big)\mathrm{d}t + \sigma \mathrm{d}W^1(t), \\ \mathrm{d}X(t) = (r(t)X(t) - Z(t)u(t))\,\mathrm{d}t + \sigma u(t)\mathrm{d}W^1(t), \\ \mathrm{d}Z(t) = k\,(\theta - Z(t))\,\mathrm{d}t + \rho\gamma \mathrm{d}W^1(t) + \sqrt{1-\rho^2}\gamma \mathrm{d}W^{\perp}(t). \end{cases} \tag{2.13}$$

Let us mention that the agent only has knowledge of the information generated by the observed commodity price; that is the information given by the filtration $\mathbb{F}^{\tilde{S}} = \{\mathcal{F}_t^{\tilde{S}}\}_{t\geq 0}$. Since the convenience yield is not observed, the above problem can be seen as a partial observation control problem from a modeling point of view.

Let us also observe the following: the drift coefficient in the dynamic of the observation process $Y(t) = \ln\tilde{S}(t)$ is affine on the unobserved factor $Z(t)$ but is independent of $Y(t)$ whereas the drift of the unobserved factor $Z(t)$ (see (2.13)) is only affine in $Z(t)$. The drift of the wealth is affine on the wealth process itself. Their diffusions are independent on the processes. In the sequel, we consider a more general model for the commodity and unobserved convenience yield prices that include the above one as a particular case. Filtering theory will then enable us to reduce the partial observation control problem (2.12)-(2.13) of systems of SDEs into a full observation control problem of a system of SDEs and SPDE.

2.2. **From partial to full information.** As already stated earlier, in this section, we use the filtering theory to transform the partial information control problem (2.12) to a full information control problem. For this purpose, we briefly summarize some known results (see for example [1, 2, 4]); in particular, we follow the exposition in [4].

In the following, we consider a general model of both the observed and unobserved factor that includes the above example. Let $W^{\perp}$ and W be two independent m-dimensional Brownian motions. Let us consider the subsequent general correlated model for observed and non-observed process Y and Z, respectively. We assume that $Y(t) = \{Y(t), t \in [0,T]\}$ and $Z(t) = \{Z(t), t \in [0,T]\}$ are n and d-dimensional processes whose dynamics are respectively given by:

$$\mathrm{d}Y(t) = h\,(t, Z(t), Y(t))\,\mathrm{d}t + \sigma\,(t, Y(t))\,\mathrm{d}W(t);\ Y(0) = 0, \tag{2.14}$$

and

$$\mathrm{d}Z(t) = b\,(t, Z(t), Y(t))\,\mathrm{d}t + \alpha\,(t, Z(t), Y(t))\,\mathrm{d}W(t) + \gamma\,(t, Z(t), Y(t))\,\mathrm{d}W^{\perp}(t);\ Z(0) = \varepsilon, \tag{2.15}$$

We further make the following assumptions (compared with [4, 8]):

Assumption A2.

- $h : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ *is globally continuous and of linear growth* (*in* z *and* y).
- $\sigma : [0,T] \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ *is uniformly continuous and has bounded* $C^3(\mathbb{R}^m)$ *-norm and satisfies the following:* $\sigma\sigma' \geq \lambda I$ *for all* y *and* t*, for some constant* $\lambda > 0$ (*uniform ellipticity condition*)*. Here* $'$ *denote the transposition.*
- $\alpha : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$ *and* $\gamma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$ *are uniformly continuous, and* α *is uniformly elliptic.*
- $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d$ *are uniformly continuous in* z *and* y *and* C^2*-bounded.*

Remark 2.1. *As pointed in [4], although our model does not have bounded drift,, one can use localization argument to take into consideration linear-growth coefficient.*

In the sequel, let $\mathcal{F}_t^Y = \sigma\{Y(s), 0 \leq s \leq t\}$ be the σ-algebra generated by the observation process $Y(t)$. The above σ-algebra is equivalent to the one generated by $\tilde{S}$. Recall that an admissible control must be adapted to $\mathcal{F}_t^Y$. Hence, in order to obtain such control, the unknown parameter $Z(t)$ is replaced by its conditional expectation with respect to $\mathcal{F}_t^Y$ in the optimal control problem (2.12).

Next, assume that $D(t) = D(t, Y(t)) := \sigma\sigma'(t, Y(t))$ is symmetric and invertible and define the process

$$\mathrm{d}\varphi(t) = -\varphi(t)h^\top(t, Z(t), Y(t))\, D^{-1/2}(t, Y(t))\,\mathrm{d}W(t),\ \ \varphi(0) = 1. \tag{2.16}$$

Here "$\top$" denote the transpose of a matrix. Under Assumption A2, since h satisfies the linear growth condition, one can show (see for example [1, Lemma 4.1.1]) that $\varphi(t)$ is a supermartingale with $E[\varphi(t)] = 1$ for all $t \in [0, T]$, that is $\varphi(t)$ is a martingale. Define the new probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_t, 0 \le t \le T$ by

$$\mathrm{d}\tilde{\mathbb{P}} := \varphi(t)\mathrm{d}\mathbb{P} \text{ on } \mathcal{F}_t, 0 \le t \le T. \tag{2.17}$$

Using Girsanov theorem, there exists a Brownian motion $\tilde{W}$ under $\tilde{\mathbb{P}}$ such that

$$\mathrm{d}Y(t) = \sigma(t, Y(t))\,\mathrm{d}\tilde{W}(t) \tag{2.18}$$

and

$$\begin{aligned}\mathrm{d}Z(t) = &\Big(b(t, Z(t), Y(t)) - \alpha^\top(t, Z(t), Y(t))\, h^\top(t, Z(t), Y(t))\, D^{-1/2}(t)\Big)\,\mathrm{d}t \\ &+ \alpha^\top(t, Z(t), Y(t))\, D^{-1/2}(t)\mathrm{d}Y(t) + \gamma(t, Z(t), Y(t))\,\mathrm{d}W^\perp(t).\end{aligned} \tag{2.19}$$

Define the process

$$\mathrm{d}\tilde{Y}(t) := D^{-1/2}(t)\mathrm{d}Y(t). \tag{2.20}$$

Then $\mathrm{d}\tilde{Y}(t)$ is a Brownian motion under $\tilde{\mathbb{P}}$. One can also show (see [1]) that $\mathrm{d}\tilde{Y}$ and $W^\perp$ are two independent Brownian motions. Moreover, since $D(t)$ is invertible, $\mathcal{F}_t^Y = \mathcal{F}_t^{\tilde{Y}}$. Define

$$\begin{aligned}K(t) = \frac{1}{\varphi(t)} :=& \exp\Big\{\int_0^t h^\top(s, Z(s), Y(s))\, D^{-1/2}(s)\mathrm{d}W(s) \\ &+\frac{1}{2}\int_0^t h^\top(s, Z(s), Y(s))\, D^{-1}(s)h(s, Z(s), Y(s))\,\mathrm{d}s\Big\} \\ =& \exp\Big\{\int_0^t h^\top(s, Z(s), Y(s))\, D^{-1}(s)\mathrm{d}Y(s) \\ &-\frac{1}{2}\int_0^t h^\top(s, Z(s), Y(s))\, D^{-1}(s)h(s, Z(s), Y(s))\,\mathrm{d}s\Big\}.\end{aligned} \tag{2.21}$$

Then $K(t)$ is a martingale. Assume that there exists a process $\Phi(t, z) = \Phi(t, z, \omega)$, $(t, z, \omega) \in [0, T] \times \mathbb{R}^d \times \Omega$ such that

$$\tilde{\mathbb{E}}\Big[f(Z(t))K(t)\Big|\mathcal{F}_t^Y\Big] = \int_{\mathbb{R}^d} f(z)\Phi(t, z)\mathrm{d}z,\ \ f \in C_0^\infty(\mathbb{R}^d), \tag{2.22}$$

where $C_0^\infty(\mathbb{R}^d)$ denotes the set of infinitely differentiable functions on $\mathbb{R}^d$ with compact support and $\tilde{\mathbb{E}}$ denotes the expectation with respect to $\tilde{\mathbb{P}}$. The process $\Phi(t, z)$ is called the *unnormalized conditional density* of $Z(t)$ given $\mathcal{F}_t^Y$.

Let L_Z denotes the second-order elliptic operator associated to $Z(t)$, then L_Z is defined by

$$L_Z := \sum_i g_i(s, z, y)\frac{\partial}{\partial z_i} + \frac{1}{2}\sum_{i,j}\Big(\alpha\alpha^\top + \gamma\gamma^\top\Big)_{i,j}(s, y, z)\frac{\partial^2}{\partial z_i \partial z_j}. \tag{2.23}$$

Denote by L^* its formal adjoint. By applying Itô's formula to $K(t)f\left(Z(t)\right)$, taking expectation and using integration by parts, one finds that the process $\Phi(t,z)$ satisfies the following Zakai equation

$$\begin{cases} \mathrm{d}\Phi(t,z) &= L^*\Phi(t,z)\mathrm{d}t + M^*\Phi(t,z)\mathrm{d}\tilde{Y}(t), \quad t\in[0,T], \\ \Phi(0,z) &= \xi(z), \end{cases} \tag{2.24}$$

where $\xi(z)$ is the density of $Z(0)$ and

$$M^*\Phi(t,z) = h(t,z,y) - \sum \frac{\partial}{\partial z_i}\left(\alpha_i(t,z,y)\cdot\Phi(t,z)\right).$$

Remark 2.2. *Assuming that the initial condition $\xi(z)$ is adapted, square integrable and smooth enough, one can show under Assumption A2 that the SPDE* (2.24) *has a unique $\mathcal{F}^Y$-adapted strong solution in an appropriate Sobolev space; see for example* [8, Proposition 2.2].

Assume in addition that the wealth process $X(t)=\{X(t),t\in[0,T]\}$ satisfies the following SDE

$$\mathrm{d}X(t) = \tilde{h}\left(t,Z(t),X(t),u(t)\right)\mathrm{d}t + \tilde{\sigma}\left(t,X(t),u(t)\right)\mathrm{d}W(t);\; X(0)=x, \tag{2.25}$$

where the coefficients $\tilde{h}$ and $\tilde{\sigma}$ are such that the above SDE has a unique strong solution. For example, such unique solution exists if the coefficients satisfy for example global Lipstichz and linear growth conditions.

Applying once more Girsanov theorem, we obtain

$$\begin{aligned} \mathrm{d}X(t) &= \Big(\tilde{h}\left(t,Z(t),X(t),u(t)\right) - \tilde{\sigma}^\top\left(t,X(t),u(t)\right)h^\top\left(t,Z(t),Y(t)\right)D^{-1/2}(t)\Big)\,\mathrm{d}t \\ &\quad + \tilde{\sigma}^\top\left(t,X(t),u(t)\right)D^{-1/2}(t)\mathrm{d}Y(t) \\ &= \Big(\tilde{h}\left(t,Z(t),X(t),u(t)\right) - \tilde{\sigma}^\top\left(t,X(t),u(t)\right)h^\top\left(t,Z(t),Y(t)\right)D^{-1/2}(t)\Big)\,\mathrm{d}t \\ &\quad + \tilde{\sigma}^\top\left(t,X(t),u(t)\right)\mathrm{d}\tilde{Y}(t). \end{aligned} \tag{2.26}$$

Combining (2.12) and (2.22), we can tranform the partial observation control problem for SDE to a full observation control problem for SPDE

$$\begin{aligned} &\sup_{u\in\mathcal{U}_{ad}} \mathbb{E}\left[\int_{\mathbb{R}} U\left(X^{x,u}(T) + \Pi\left(\exp\{Y(T)\}, B\left(Z(T)+\bar{b}\right)\right)\right)\mathrm{d}\mathbb{P}_{\bar{B}}\right] \\ &= \sup_{u\in\mathcal{U}_{ad}} \tilde{\mathbb{E}}\left[\int_{\mathbb{R}}\int_{\mathbb{R}^d} U\left(X^{x,u}(T) + \Pi\left(\exp\{Y(T)\}, B\left(z+\bar{b}\right)\right)\right)\mathrm{d}\mathbb{P}_{\bar{B}}\Phi(T,z)\mathrm{d}z\right], \end{aligned} \tag{2.27}$$

where $X(t)$ and $\Phi(t,z)$ are given by (2.26) and (2.24), respectively. Here $\tilde{S}(t)=\exp\{Y(t)\}$ is given by

$$\mathrm{d}\tilde{S}(t) = \tilde{S}(t)\Big(\frac{1}{2}D(t)\mathrm{d}t + D^{1/2}(t)\mathrm{d}\tilde{Y}(t)\Big).$$

Note that the control only affects the wealth process $X^{x,u}$ and not the commodity price process $\tilde{S}(T)$ nor the density $\Phi(t,z)$. We summarize the full observation counterpart of the model described in Section 2.1 in the following remark.

Remark 2.3. *In our model* $Y = \log \tilde{S}$ *and* $\mathrm{d}\tilde{Y}(t) = D^{-1/2}(t)\mathrm{d}Y(t) = \frac{1}{\sigma}\mathrm{d}S(t)$*. It follows that*

$$\begin{cases} \mathrm{d}\tilde{S}(t) = \frac{1}{2}\sigma^2\tilde{S}(t)\mathrm{d}t + \tilde{S}(t)\sigma\mathrm{d}\tilde{Y}(t), \\ \mathrm{d}X^{x,u}(t) = \Big(r(t)X(t) - (r(t) - \frac{1}{2}\sigma^2)u(t)\Big)\mathrm{d}t + u(t)\sigma\mathrm{d}\tilde{Y}(t), \\ \mathrm{d}\Phi(t,z) = \Big(\frac{1}{2}\gamma^2\frac{\partial^2\Phi(t,z)}{\partial z^2} + \frac{\partial}{\partial z}\left(k(\theta - z)\Phi(t,z)\right)\Big)\mathrm{d}t + \Big(r(t) - \frac{1}{2}\sigma^2 - z - \rho\gamma\frac{\partial\Phi(t,z)}{\partial z}\Big)\mathrm{d}\tilde{Y}(t), \end{cases} \tag{2.28}$$

where $\tilde{Y}(t)$ *is a standard Brownian motion under* $\tilde{\mathbb{P}}$*.*

Define $\mathfrak{L}\Phi(t,z) := \dfrac{\gamma^2}{2}\dfrac{\partial^2}{\partial z^2}\Phi(t,z)$ and $b\left(t,z,\Phi(t,z),\Phi'(t,z)\right) := -k\Phi(t,z) + k\left(\theta - z\right)\Phi'(t,z)$ so that

$$L^*\Phi(t,z) = \mathfrak{L}\Phi(t,z) + b\Big(t,z,\Phi(t,z),\frac{\partial\Phi(t,z)}{\partial z}\Big) \tag{2.29}$$

and define $M^*\Phi(t,z) = \sigma\Big(t,z,\Phi(t,z),\dfrac{\partial\Phi(t,z)}{\partial z}\Big) := r^2(t) - \dfrac{1}{2}\sigma^2 - z - \rho\gamma\dfrac{\partial\Phi(t,z)}{\partial z}$. Then we obtain

$$\begin{aligned} \mathrm{d}\Phi(t,z) = &\Big\{\mathfrak{L}\Phi(t,z) + b\Big(t,z,\Phi(t,z),\frac{\partial\Phi(t,z)}{\partial z}\Big)\Big\}\mathrm{d}t \\ &+ \sigma\Big(t,z,\Phi(t,z),\frac{\partial\Phi(t,z)}{\partial z}\Big)\mathrm{d}\tilde{Y}(t), \quad t \in [0,T]. \end{aligned} \tag{2.30}$$

Let us observe the following: in the above SDEs for S and X, the coefficients are affine in their parameters. The drift coefficient of the SPDE depends on a linear differential operator, whereas its diffusion coefficient is affine in the first order derivative of the SPDE. In the next section, we use a model that has the above one as a particular case and present general sufficient and equivalent stochastic maximum principles to the above optimal control problem (2.27).

3. Stochastic Maximum Principle for factor models

In this section, we consider a more general framework. We assume a more general form of the processes $X(t), Y(t)$ and $\Phi(t,z)$. We first derive sufficient maximum principle for the optimal control (2.12)-(2.30). Second, we derive an equivalent maximum principle.

Let $T > 0$, be a fixed time horizon. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be a filtered probability space on which is given a one dimensional standard Brownian motion $W(t)$. In the previous section setting, this probability space corresponds to $\Big(\Omega, \mathcal{F}, \{\mathcal{F}_t^{\tilde{Y}}\}_{t\in[0,T]}, \tilde{\mathbb{P}}\Big)$ with the Brownian motion $\tilde{Y}$. For clarity of the exposition, we work in one dimension, extension to the multidimensional case follows similarly. The state process is defined by the triplet $(Y(t), X(t), \Phi(t,z))$ whose dynamics are respectively given by:

$$\mathrm{d}Y(t) = b_1\left(t, Y(t), u(t)\right)\mathrm{d}t + \sigma_1\left(t, Y(t), u(t)\right)\mathrm{d}W(t), \ \ Y(0) = y_0, \tag{3.1}$$

$$\mathrm{d}X(t) = b_2\left(t, X(t), u(t)\right)\mathrm{d}t + \sigma_2\left(t, X(t), u(t)\right)\mathrm{d}W(t), \ \ X(0) = x_0, \tag{3.2}$$

$$\begin{cases} \mathrm{d}\Phi(t,z) = \Big(L\Phi(t,z) + b_3\Big(t,z,\Phi(t,z),\frac{\partial\Phi(t,z)}{\partial z},u(t)\Big)\Big)\,\mathrm{d}t \\ \qquad\qquad + \sigma_3\Big(t,z,\Phi(t,x),\frac{\partial\Phi(t,z)}{\partial z},u(t,z)\Big)\,\mathrm{d}W(t) \\ \Phi(0,z) = \xi(z); z\in\mathbb{R} \\ \lim_{\|z\|\to\infty}\Phi(t,z) = 0, t\in[0,T], \end{cases} \tag{3.3}$$

where L is a linear differential operator acting on x; $b_1, b_2, b_3, \sigma_1, \sigma_2, \sigma_3$ are given functions satisfying conditions of existence and uniqueness of strong solution of the system (3.1)-(3.3); see for example [4, Lemma 4.1] (see also [8, 12, 13, 14, 26]) for (3.3) and [10, 21] for (3.1)-(3.2)). Let f and g be given C^1 functions with respect to their arguments. We define

$$\begin{aligned} J(u) =& \mathbb{E}\Big[\int_{\mathbb{R}}\Big[\int_0^T\int_{\mathbb{R}} f\left(t,z,X(t),Y(t),\Phi(t,z),\bar{b},u(t)\right)\mathrm{d}z\mathrm{d}t \\ &+ \int_{\mathbb{R}} g\left(z,X(T),Y(T),\Phi(T,z),\bar{b}\right)\mathrm{d}z\Big]\,\mathrm{d}\mathbb{P}_{\bar{B}}\Big]. \end{aligned} \tag{3.4}$$

We denote by $\mathcal{U}_{ad}$ the set of admissible controls contained in the set of $\mathcal{F}_t$-predictable control such that the system (3.1)-(3.3) has a unique strong solution and

$$\begin{aligned} &\mathbb{E}\Big[\int_{\mathbb{R}}\Big[\int_0^T\int_{\mathbb{R}}\Big|f\left(t,z,X(t),Y(t),\Phi(t,z),\bar{b},u(t)\right)\Big|\mathrm{d}z\mathrm{d}t \\ &+\int_{\mathbb{R}}\Big|g\left(z,X(T),Y(T),\Phi(T,z),\bar{b}\right)\Big|\mathrm{d}z\Big]\mathrm{d}\mathbb{P}_{\bar{B}}\Big] < \infty. \end{aligned}$$

We are interested in the following control problem

Problem 3.1. *Find the maximizer $\hat{u}$ of J, that is find $\hat{u}\in\mathcal{U}_{ad}$ such that*

$$J(\hat{u}) = \sup_{u\in\mathcal{U}_{ad}} J(u). \tag{3.5}$$

3.1. **Sufficient stochastic maximum principle.** We first define the Hamiltonian $H:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times U\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned} H\left(t,z,x,y,\phi,\phi',u,p_1,q_1,p_2,q_2,p_3,q_3\right) =& \int_{\mathbb{R}} f\left(t,z,x,\phi,\bar{b},u\right)\mathrm{d}\mathbb{P}_{\bar{B}} + b_1\left(t,y,u\right)p_1 + \sigma_1\left(t,y,u\right)q_1 \\ &+ b_2\left(t,x,u\right)p_2 + \sigma_2\left(t,x,u\right)q_2 \\ &+ b_3\left(t,z,\phi,\phi'\right)p_3 + \sigma_3\left(t,z,\phi,\phi'\right)q_3, \end{aligned} \tag{3.6}$$

where $\phi' = \dfrac{\partial\phi}{\partial z}$. Suppose that H is differentiable in the variable x, y, ϕ and ϕ'. For $u\in\mathcal{U}_{ad}$, we consider the adjoint processes satisfying the system of backward stochastic (partial) differential equations in the unknowns $p_1(t,z), q_1(t,z), p_2(t,z), q_2(t,z), p_3(t,z), q_3(t,z) \in \mathbb{R}$

$$
\begin{cases}
\mathrm{d}p_1(t,z) & = & -\dfrac{\partial H(t,z)}{\partial y}\mathrm{d}t + q_1(t,z)\mathrm{d}W(t) \\
p_1(T,z) & = & \displaystyle\int_{\mathbb{R}} \dfrac{\partial g(z,\bar{b})}{\partial y}\mathrm{d}\mathbb{P}_{\bar{B}} \\
\mathrm{d}p_2(t,z) & = & -\dfrac{\partial H(t,z)}{\partial x}\mathrm{d}t + q_2(t,z)\mathrm{d}W(t) \\
p_2(T,z) & = & \displaystyle\int_{\mathbb{R}} \dfrac{\partial g(z,\bar{b})}{\partial x}\mathrm{d}\mathbb{P}_{\bar{B}} \\
\mathrm{d}p_3(t,z) & = & -\Big(L^* p_3(t,z) + \dfrac{\partial H\,(t,z)}{\partial \phi} - \dfrac{\partial}{\partial z}\Big(\dfrac{\partial H\,(t,z)}{\partial \phi'}\Big)\mathrm{d}t + q_3(t,z)\mathrm{d}W(t) \\
p_3(T,z) & = & \displaystyle\int_{\mathbb{R}} \dfrac{\partial g(z,\bar{b})}{\partial \phi}\mathrm{d}\mathbb{P}_{\bar{B}} \\
\lim\limits_{\|z\|\to\infty} p_3(T,z) & = & 0,
\end{cases}
\tag{3.7}
$$

where L^* is the adjoint of L and we have used the short hand notation
$g(z) = g(z, X(T), Y(T), \Phi(T,z), \bar{b})$ and

$$H\,(t,z) = H\,\big(t, z, X(t), Y(t), u(t), \Phi(t,z), \Phi'(t,z), p_1(t,z), q_1(t,z), p_2(t,z), q_2(t,z), p_3(t,z), q_3(t,z)\big)\,.$$

Remark 3.2. *If one assumes for example that the coefficients of the controlled processes, the profit rate and the bequest functions of the performance functional are smooth enough, then there exists a unique strong classical solution for the system of BSDEs and BSPDE representing the associated adjoint processes; see for example* [5, 11] *and references therein.*

Next we give the sufficient stochastic maximum principle.

Theorem 3.3 (Sufficient stochastic maximum principle)**.** *Let $\hat{u} \in \mathcal{U}_{ad}$ with corresponding solutions $\hat{Y}(t), \hat{X}(t), \hat{\Phi}(t,z)$, $(\hat{p_1}(t,z), \hat{q_1}(t,z)); (\hat{p_2}(t,z), \hat{q_2}(t,z)); (\hat{p_3}(t,x), \hat{q_3}(t,x))$ of* (3.1)-(3.7)*. Suppose that the followings hold:*

(i) *The function $(x,y,\phi) \mapsto g(z,x,y,\phi)$ is a concave function of x, y, ϕ for all $z \in \mathbb{R}$.*

(ii) *The function*

$$\widetilde{h}(x,y,\phi,\phi') = \sup_{u\in\mathcal{U}_{ad}} H\,\big(t, z, x, y, u, \phi, \phi', \widehat{p}_1(t,z), \widehat{q}_1(t,z), \widehat{p}_2(t,z), \widehat{q}_2(t,z), \widehat{p}_3(t,z), \widehat{q}_3(t,z)\big)\,. \tag{3.8}$$

exists and is a concave function of x, y, ϕ, ϕ' for all $(t,z) \in [0,T] \times \mathbb{R}$ a.s.

(iii) (The maximum condition)

$$
\begin{aligned}
&H\,\Big(t, z, \hat{X}(t), \hat{Y}(t), \hat{u}(t), \hat{\Phi}(t,z), \hat{\Phi}'(t,z), \widehat{p}_1(t,z), \widehat{q}_1(t,z), \widehat{p}_2(t,z), \widehat{q}_2(t,z), \widehat{p}_3(t,z), \widehat{q}_3(t,z)\Big) \\
= &\sup_{v\in\mathcal{U}_{ad}} H\,\Big(t, z, \hat{X}(t), \hat{Y}(t), v, \hat{\Phi}(t,z), \hat{\Phi}'(t,z), \widehat{p}_1(t,z), \widehat{q}_1(t,z), \widehat{p}_2(t,z), \widehat{q}_2(t,z), \widehat{p}_3(t,z), \widehat{q}_3(t,z)\Big)\,.
\end{aligned}
\tag{3.9}
$$

(iv) *Assume in addition that the following integral conditions hold*

$$\mathbb{E}\left[\int_{\mathbb{R}}\int_0^T \left\{\Big(\Phi(t,z) - \hat{\Phi}(t,z)\Big)^2 \hat{q}_3^2(t,z) + \hat{p}_3^2(t,z)\sigma_3^2(t,z,\Phi(t,z),\Phi'(t,z),u(t))\right\}\mathrm{d}t\mathrm{d}z\right] < \infty$$

and

$$\mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T \Big\{\Big(X(t)-\hat{X}(t)\Big)^2\hat{q}_1^2(t,z)^2+\hat{p}_1^2(t,z)\sigma_1^2(t,X(t),u(t)) + \Big(Y(t)-\hat{Y}(t)\Big)^2\hat{q}_2^2(t,z)+\hat{p}_2^2(t,z)\sigma_2^2(t,Y(t),u(t))\Big\}\mathrm{d}t\mathrm{d}z\Big]<\infty$$

for all $u\in\mathcal{U}_{ad}$.

Then $\hat{u}(t)$ *is an optimal control for the control problem* (3.1)-(3.5).

Proof. We will prove that $J(\hat{u})\ge J(u)$ for all $u\in\mathcal{U}_{ad}$. Choose $u\in\mathcal{U}_{ad}$ and let $X(t)=X^u(t), Y(t)=Y^u(t)$ and $\Phi(t,z)=\Phi^u(t,Z)$ be the corresponding solutions to (3.1)-(3.3). In the sequel, we use the short hand notation:

$$\begin{aligned}
b_1(t)=&b_1\left(t,Y(t),u(t)\right),\ \hat{b}_1(t)=b_1(t,\hat{Y}(t),\hat{u}(t)),\\
\sigma_1(t)=&\sigma_1\left(t,Y(t),u(t)\right),\ \hat{\sigma}_1(t)=\sigma_1(t,\hat{Y}(t),\hat{u}(t)),\\
b_2(t)=&b_2\left(t,Y(t),u(t)\right),\ \hat{b}_2(t)=b_2(t,\hat{Y}(t),\hat{u}(t)),\\
\sigma_2(t)=&\sigma_2\left(t,Y(t),u(t)\right),\ \hat{\sigma}_2(t)=\sigma_2(t,\hat{Y}(t),\hat{u}(t)),\\
b_3(t,z)=&b_3(t,z,\Phi(t,z),\Phi'(t,z),u(t)),\ \hat{b}_3(t,z)=\hat{b}_3(t,z,\hat{\Phi}(t,z),\hat{\Phi}'(t,z),\hat{u}(t)),\mathrm{etc}
\end{aligned}$$

Since $\int_{\mathbb{R}} f(t,z,\bar{b})\mathrm{d}\mathbb{P}_{\bar{B}}$ does not depend on $\hat{p_1}(t,x),\hat{q_1}(t,x),\hat{p_2}(t,z),\hat{q_2}(t,z)),\hat{p_3}(t,z)$ and $\hat{q_3}(t,z)$, we can write

$$\begin{aligned}
\int_{\mathbb{R}}\hat{f}(t,z,\bar{b})\mathrm{d}\mathbb{P}_{\bar{B}}=&\hat{H}(t,z)-\hat{b}_1(t)\hat{p}_1(t,z)-\hat{\sigma}_1(t)\hat{q}_1(t,z)-\hat{b}_2(t)\hat{p}_2(t,z)-\hat{\sigma}_2(t)\hat{q}_2(t,z)\\
&-\hat{b}_3(t,z)\hat{p}_3(t,z)-\hat{\sigma}_3(t,z)\hat{q}_3(t,z)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} f(t,z,\bar{b})\mathrm{d}\mathbb{P}_{\bar{B}}=&H(t,z)-b_1(t)\hat{p}_1(t,z)-\sigma_1(t)\hat{q}_1(t,z)-b_2(t)\hat{p}_2(t,z)-\sigma_2(t)\hat{q}_2(t,z)\\
&-b_3(t,z)\hat{p}_3(t,z)-\sigma_3(t,z)\hat{q}_3(t,z).
\end{aligned}$$

Using the above and (3.6), we have

$$\begin{aligned}
J(\hat{u})-J(u)=&\mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}}(\hat{f}(t,z,\bar{b})-f(t,z,\bar{b}))\mathrm{d}z\mathrm{d}t\mathrm{d}\mathbb{P}_{\bar{B}}\Big]+\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}}(\hat{g}(z,\bar{b})-g(z,\bar{b}))\mathrm{d}z\mathrm{d}\mathbb{P}_{\bar{B}}\Big]\\
=&I_1+I_2,
\end{aligned}\tag{3.10}$$

with

$$\begin{aligned}
I_1=&\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}\Big\{\hat{H}(t,z)-H(t,z)-\Big(\hat{b}_1(t)-b_1(t)\Big)\hat{p}_1(t)-\Big(\hat{\sigma}_1(t)-\sigma_1(t)\Big)\hat{q}_1(t)\\
&-\Big(\hat{b}_2(t)-b_2(t)\Big)\hat{p}_2(t)-\Big(\hat{\sigma}_2(t)-\sigma_2(t)\Big)\hat{q}_2(t)\\
&-\Big(\hat{b}_3(t,z)-b_3(t,z)\Big)\hat{p}_3(t)-\Big(\hat{\sigma}_3(t,z)-\sigma_3(t,z)\Big)\hat{q}_3(t)\Big\}\mathrm{d}z\mathrm{d}t\Big],\\
I_2=&\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}}(\hat{g}(z,\bar{b})-g(z,\bar{b}))\mathrm{d}z\mathrm{d}\mathbb{P}_{\bar{B}}\Big].
\end{aligned}$$

Now, using the concavity of $(x,y,\phi)\mapsto g(z,x,y,\phi)$ and the Itô's formula, we get

$$\begin{aligned}
I_2 \geq &\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}}\Big\{\frac{\partial \hat{g}(z,\bar{b})}{\partial x}\Big(\hat{X}(T)-X(T)\Big)+\frac{\partial \hat{g}(z,\bar{b})}{\partial y}\Big(\hat{Y}(T)-Y(T)\Big)\\
&+\frac{\partial \hat{g}(z,\bar{b})}{\partial \phi}\Big(\hat{\Phi}(T,z)-\Phi(T,z)\Big)\Big\}\mathrm{d}z\mathrm{d}\mathbb{P}_{\bar{B}}\Big]\\
=&\mathbb{E}\left[\int_{\mathbb{R}}\left\{\hat{p}_1(T,z)\left(\hat{X}(T)-X(T)\right)+\hat{p}_2(T,z)\left(\hat{Y}(T)-Y(T)\right)\right.\right.\\
&\left.\left.+\hat{p}_3(T,z)\left(\hat{\Phi}(T,z)-\Phi(T,z)\right)\right\}\mathrm{d}z\right]\\
=&\mathbb{E}\left[\int_{\mathbb{R}}\left\{\hat{p}_1(0,z)\left(\hat{X}(0)-X(0)\right)+\int_0^T\left(\hat{X}(t)-X(t)\right)\mathrm{d}p_1(t,z)\right.\right.\\
&+\int_0^T\hat{p}_1(t,z)\mathrm{d}\left(\hat{X}(t)-X(t)\right)+\int_0^T\hat{q}_1(t,z)\left(\hat{\sigma}_1(t)-\sigma_1(t)\right)\mathrm{d}t\\
&+\hat{p}_2(0,z)\left(\hat{Y}(0)-Y(0)\right)+\int_0^T\left(\hat{Y}(t)-Y(t)\right)\mathrm{d}p_2(t,z)+\int_0^T\hat{p}_2(t,z)\mathrm{d}\left(\hat{Y}(t)-Y(t)\right)\\
&+\int_0^T\hat{q}_2(t,z)\left(\hat{\sigma}_2(t)-\sigma_2(t)\right)\mathrm{d}t+\hat{p}_3(0,x)\left(\hat{\Phi}(0,x)-\Phi(0,x)\right)\\
&+\int_0^T\left(\hat{\Phi}(t,z)-\Phi(t,z)\right)\mathrm{d}p_3(t,z)+\int_0^T\hat{p}_3(t,z)\mathrm{d}\left(\hat{\Phi}(t,z)-\Phi(t,z)\right)\\
&\left.\left.+\int_0^T\hat{q}_3(t,x)\left(\hat{\sigma}_3(t,z)-\sigma_3(t,z)\right)\mathrm{d}t\right\}\mathrm{d}z\right]\\
\geq&\mathbb{E}\Big[\int_{\mathbb{R}}\Big\{\int_0^T-\left(\hat{X}(T)-X(T)\right)\frac{\partial\hat{H}(t,z)}{\partial x}\mathrm{d}t+\int_0^T\hat{p}_1(t,z)\left(\hat{b}_1(t)-b_1(t)\right)\mathrm{d}t\\
&+\int_0^T\hat{q}_2(t,z)\left(\hat{\sigma}_2(t)-\sigma_2(t)\right)\mathrm{d}t-\int_0^T\left(\hat{Y}(T)-Y(T)\right)\frac{\partial\hat{H}(t,z)}{\partial y}\mathrm{d}t\\
&+\int_0^T\hat{p}_2(t,z)\Big(\hat{b}_2(t)-b_2(t)\Big)\mathrm{d}t+\int_0^T\hat{q}_2(t,z)\Big(\hat{\sigma}_2(t)-\sigma_2(t)\Big)\mathrm{d}t\\
&-\int_0^T\left(\hat{\Phi}(t,z)-\Phi(t,z)\right)\left(L^*\hat{p}_3(t,z)+\frac{\partial\hat{H}(t,z)}{\partial\phi}-\frac{\partial}{\partial z}\Big(\frac{\partial\hat{H}(t,z)}{\partial\phi'}\Big)\right)\mathrm{d}t\\
&+\int_0^T\hat{p}_3(t,z)\Big(L\Big(\hat{\Phi}(t,z)-\Phi(t,z)\Big)\Big)+\Big(\hat{b}_3(t,z)-b_3(t,z)\Big)\mathrm{d}t\\
&+\int_0^T\hat{q}_3(t,z)\left(\hat{\sigma}_3(t,z)-\sigma_3(t,z)\right)\mathrm{d}t\Big\}\mathrm{d}z\Big].
\end{aligned}\tag{3.11}$$

Since $\lim_{\|z\|\to\infty}\left(\hat{\Phi}(t,z)-\Phi(t,z)\right)=\lim_{\|z\|\to 0}\hat{p}_3(T,x)=0$, we have

$$\int_{\mathbb{R}}\left(\hat{\Phi}(t,z)-\Phi(t,z)\right)L^*\hat{p}_3(t,z)\mathrm{d}z=\int_{\mathbb{R}}\hat{p}_3(t,z)L\left(\hat{\Phi}(t,z)-\Phi(t,z)\right)\mathrm{d}z. \tag{3.12}$$

Combining (3.10), (3.11) and (3.12) we get

$$\begin{aligned}
J(\hat{u})-J(u) \geq & \mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T \Big\{\Big(\hat{H}(t,z)-H(t,z)\Big)-\frac{\partial \hat{H}(t,z)}{\partial x}\Big(\hat{X}(t)-X(t)\Big) \\
& -\frac{\partial \hat{H}(t,z)}{\partial y}\Big(\hat{Y}(t)-Y(t)\Big) \\
& -\Big(\frac{\partial \hat{H}(t,z)}{\partial \phi}-\frac{\partial}{\partial z}\Big(\frac{\partial \hat{H}(t,z)}{\partial \phi'}\Big)\Big)\Big(\hat{\Phi}(t,z)-\Phi(t,z)\Big)\Big\}\mathrm{d}t\mathrm{d}z\Big] \\
= & \mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T \Big\{\Big(\hat{H}(t,z)-H(t,z)\Big)-\frac{\partial \hat{H}(t,z)}{\partial x}\Big(\hat{X}(t)-X(t)\Big) \\
& -\frac{\partial \hat{H}(t,z)}{\partial y}\Big(\hat{Y}(t)-Y(t)\Big)-\frac{\partial \hat{H}(t,z)}{\partial \phi}\Big(\hat{\Phi}(t,z)-\Phi(t,z)\Big) \\
& -\frac{\partial \hat{H}(t,z)}{\partial \phi'}\Big(\frac{\partial \hat{\Phi}(t,z)}{\partial z}-\frac{\partial \Phi(t,z)}{\partial z}\Big)\Big\}\mathrm{d}t\mathrm{d}z\Big]. \qquad (3.13)
\end{aligned}$$

One can show, using the same arguments in [6] that, the right hand side of (3.13) is non-negative. For sake of completeness we shall give the details here. Fix $t \in [0,T]$. Since $\widetilde{h}(x,y,\phi,\phi')$ is concave in x,y,ϕ,ϕ', it follows by the standard hyperplane argument that (see e.g [24, Chapter 5, Section 23]) there exists a subgradient $d=(d_1,d_2,d_3,d_4) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ for $\widetilde{h}(x,y,\phi,\phi')$ at $x=\hat{X}(t)$, $y=\hat{Y}(t)$, $\phi=\widehat{\Phi}(t,x)$, $\phi'=\widehat{\Phi}'(t,x)$ such that if we define i by

$$\begin{aligned}
i(x,y,\phi,\phi') := & \widetilde{h}(x,y,\phi,\phi')-\hat{H}(t,z)-d_1(x-\hat{X}(t))-d_2(y-\hat{Y}(t)) \\
& d_3(\phi-\widehat{\Phi}(t,x))-d_4(\phi'-\widehat{\Phi}'(t,x)), \qquad (3.14)
\end{aligned}$$

then $i\Big(\hat{X}(t),\hat{Y}(t),\hat{\Phi}(t,x),\hat{\Phi}'(t,x)\Big)=0$ for all X,Y,Φ,Φ'.

It follows that,

$$\begin{aligned}
d_1 = & \frac{\partial \widetilde{h}}{\partial x}(\hat{X}(t),\hat{Y}(t)\widehat{\Phi}(t,x),\widehat{\Phi}'(t,x)), \\
d_2 = & \frac{\partial \widetilde{h}}{\partial y}(\hat{X}(t),\hat{Y}(t)\widehat{\Phi}(t,x),\widehat{\Phi}'(t,x)), \\
d_3 = & \frac{\partial \widetilde{h}}{\partial \Phi}(\hat{X}(t),\hat{Y}(t)\widehat{\Phi}(t,x),\widehat{\Phi}'(t,x)), \\
d_4 = & \frac{\partial \widetilde{h}}{\partial \Phi'}(\hat{X}(t),\hat{Y}(t)\widehat{\Phi}(t,x),\widehat{\Phi}'(t,x)).
\end{aligned}$$

Substituting this into (3.13), using conditions (ii) and (iii), we conclude that $J(\widehat{u}) \geq J(u)$ for all $u \in \mathcal{U}_{ad}$. This completes the proof. □

In the next section, we present an equivalent maximum principle which does not require the concavity assumption.

3.2. **Equivalent stochastic maximum principle.** The concavity assumption sometimes fail to be satisfied in some interesting applications. In this case one may need an equivalent maximum principle to overcome this difficulty. In order to derive such maximum principle, we need the following additional conditions

(C1) The functions $b_1, b_2, b_3, \sigma_1, \sigma_2, \sigma_3, f$ and g are C^3 with respect to their arguments x, y, Φ, u.

(C2) For all $0 < t \leq r < T$ all bounded $\mathcal{F}_t$-measurable random variables α, and all bounded, deterministic function $\zeta : \mathbb{R} \mapsto \mathbb{R}$, the control

$$\beta(s,z) = \alpha(\omega)\chi_{[t,r]}(s)\zeta(z), 0 \leq s \leq T \text{ and } (s,z) \in \Omega \times \mathbb{R} \tag{3.15}$$

belongs to $\mathcal{U}_{ad}$.

(C3) For all $u \in \mathcal{U}_{ad}$ and all bounded $\beta \in \mathcal{U}_{ad}$, there exists $r > 0$ such that

$$u + \delta\beta \in \mathcal{U}_{ad} \tag{3.16}$$

for all $\delta \in (-r, r)$ and such that the family

$$\begin{aligned}\Big\{&\frac{\partial f}{\partial x}\Big(t, z, X^{u+\delta\beta}(t), Y^{u+\delta\beta}(t), \Phi^{u+\delta\beta}(t,z), b_1, u(t,z) + \delta\beta(t,z), \omega\Big)\frac{\mathrm{d}}{\mathrm{d}\delta}X^{u+\delta\beta}(t)\\ &+\frac{\partial f}{\partial y}\Big(t, z, X^{u+\delta\beta}(t), Y^{u+\delta\beta}(t), \Phi^{u+\delta\beta}(t,z), b_1, u(t,z) + \delta\beta(t,z), \omega\Big)\frac{\mathrm{d}}{\mathrm{d}\delta}Y^{u+\delta\beta}(t)\\ &+\frac{\partial f}{\partial \phi}\Big(t, z, X^{u+\delta\beta}(t), Y^{u+\delta\beta}(t), \Phi^{u+\delta\beta}(t,z), b_1, u(t,z) + \delta\beta(t,z), \omega\Big)\frac{\mathrm{d}}{\mathrm{d}\delta}\Phi^{u+\delta\beta}(t,z)\\ &+\frac{\partial f}{\partial u}\Big(t, z, X^{u+\delta\beta}(t), Y^{u+\delta\beta}(t), \Phi^{u+\delta\beta}(t,z), u(t,z) + \delta\beta(t,z), \omega\Big)\beta(t,z)\Big\}_{\delta\in(-r,r)}\end{aligned}$$

is $\lambda \times \mathbb{P} \times \mu$-uniformly integrable;

$$\begin{aligned}\Big\{&\frac{\partial g}{\partial x}\Big(z, X^{u+\delta\beta}(T), Y^{u+\delta\beta}(T), \Phi^{u+\delta\beta}(T,z)\Big)\frac{\mathrm{d}}{\mathrm{d}\delta}X^{u+\delta\beta}(t)\\ &+\frac{\partial g}{\partial y}\Big(z, X^{u+\delta\beta}(T), Y^{u+\delta\beta}(T), \Phi^{u+\delta\beta}(T,z)\Big)\frac{\mathrm{d}}{\mathrm{d}\delta}Y^{u+\delta\beta}(t)\\ &+\frac{\partial g}{\partial \phi}\Big(z, X^{u+\delta\beta}(T), Y^{u+\delta\beta}(T), \Phi^{u+\delta\beta}(T,z)\Big)\frac{\mathrm{d}}{\mathrm{d}\delta}\Phi^{u+\delta\beta}(t,z)\Big\}_{\delta\in(-r,r)}\end{aligned}$$

is $\mathbb{P} \times \mu$-uniformly integrable.

(C4) For all $u, \beta \in \mathcal{U}_{ad}$ with β bounded, the processes

$$\begin{aligned}\Gamma_1(t) = \Gamma_1^\beta(t) &= \frac{\mathrm{d}}{\mathrm{d}\delta}Y^{u+\delta\beta}(t)\Big|_{\delta=0},\\ \Gamma_2(t) = \Gamma_2^\beta(t) &= \frac{\mathrm{d}}{\mathrm{d}\delta}X^{u+\delta\beta}(t)\Big|_{\delta=0},\\ \Gamma_3(t,z) = \Gamma_3^\beta(t) &= \frac{\mathrm{d}}{\mathrm{d}\delta}\Phi^{u+\delta\beta}(t,z)\Big|_{\delta=0},\end{aligned}$$

exist and

$$\begin{aligned}L\Gamma_3(t,z) &= \frac{\mathrm{d}}{\mathrm{d}\delta}L\Phi^{u+\delta\beta}(t,z)\Big|_{\delta=0},\\ \frac{\partial\Gamma_3(t,z)}{\partial z} &= \frac{\mathrm{d}}{\mathrm{d}\delta}\Big(\frac{\partial\Phi^{u+\delta\beta}(t,z)}{\partial z}\Big)\Big|_{\delta=0}.\end{aligned}$$

Moreover, the processes $\Gamma_1(t), \Gamma_2(t), \Gamma_3(t,z)$ satisfy

$$\mathrm{d}\Gamma_1(t) = \Big(\frac{\partial b_1(t)}{\partial y}\Gamma_1(t) + \frac{\partial b_1(t)}{\partial u}\beta(t,z)\Big)\mathrm{d}t + \Big(\frac{\partial\sigma_1(t)}{\partial y}\Gamma_1(t) + \frac{\partial\sigma_1(t)}{\partial u}\beta(t,z)\Big)\mathrm{d}W(t), \tag{3.17}$$

$$\mathrm{d}\Gamma_2(t) =\Big(\frac{\partial b_2(t)}{\partial x}\Gamma_2(t)+\frac{\partial b_2(t)}{\partial u}\beta(t,z)\Big)\mathrm{d}t+\Big(\frac{\partial \sigma_2(t)}{\partial y}\Gamma_2(t)+\frac{\partial \sigma_2(t)}{\partial u}\beta(t,z)\Big)\mathrm{d}W(t), \tag{3.18}$$

$$\begin{aligned}\mathrm{d}\Gamma_3(t,z) =&\Big(L\Gamma_3(t,z)+\frac{\partial b_3(t,z)}{\partial \phi}\Gamma(t,z)+\frac{\partial \Gamma_3(t,z)}{\partial z}\frac{\partial b_3(t,z)}{\partial \phi'}+\frac{\partial b_3(t,z)}{\partial u}\beta(t,z)\Big)\mathrm{d}t\\ &+\Big(\frac{\partial \sigma_3(t,z)}{\partial \phi}\Gamma_3(t,z)+\frac{\partial \Gamma_3(t,z)}{\partial z}\frac{\partial \sigma_3(t,z)}{\partial \phi'}+\frac{\partial \sigma_3(t,z)}{\partial u}\beta(t,z)\Big)\mathrm{d}W(t),\end{aligned} \tag{3.19}$$

with

$$\Gamma_1(0)=0,\ \Gamma_2(t)=0,\ \Gamma_3(0,z)=0 \text{ for all } z \text{ and } \lim_{\|z\|\to\infty}\Gamma_3(t,z)=0,\ t\in[0,T],$$

where we used the short hand notation

$$b_1(t)=b_1\left(t,Y(t),u(t)\right),\ \sigma_1(t)=\sigma_1\left(t,Y(t),u(t)\right),\ \text{etc.}$$

We have the following theorem

Theorem 3.4 (Equivalent stochastic maximum principle)**.** *Retain conditions* (C1)-(C4). *Let* $u\in\mathcal{U}_{ad}$ *with corresponding solutions* $X(t),Y(t),\Phi(t,z)$, $(p_1(t,z),q_1(t,z)),(p_2(t,z),q_2(t,z));(p_3(t,z),q_3(t,z)),\Gamma_1(t),\Gamma_2(t)$ *and* $\Gamma_3(t,z)$ *of* (3.1)-(3.3)*;* (3.7)*;*(3.17)-(3.19)*. Under some integrability conditions that guaranty the use of the Itô's product rules, the following are equivalent:*

(i)

$$\frac{\mathrm{d}}{\mathrm{d}s}J(u+s\beta)\Big|_{s=0}=0 \textit{ for all bounded } \beta\in\mathcal{U}_{ad}. \tag{3.20}$$

(ii)

$$\frac{\partial H}{\partial u}\big(t,z,X(t),Y(t),u(t),\Phi(t,z),\Phi'(t,z),p_1(t,z),q_1(t,z),p_2(t,z),q_2(t,z),p_3(t,z),q_3(t,z)\big)=0 \tag{3.21}$$

for all $t\in[0,T]$ *and almost all* $z\in\mathbb{R}$.

Proof.

(i) $\Rightarrow$ **(ii)**. Assume that $\frac{\mathrm{d}}{\mathrm{d}s}J(u+s\beta)\Big|_{s=0}=0$. Then

$$\begin{aligned}0=&\frac{\mathrm{d}}{\mathrm{d}s}J(u+s\beta)\Big|_{s=0}\\ =&\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}\Big[\int_{\mathbb{R}}\Big\{\int_0^T\int_{\mathbb{R}}f\Big(t,X(t),Y(t),\Phi(t,z),\bar{b},u(t,z)+s\beta(t,z)\Big)\mathrm{d}z\mathrm{d}t\\ &+\int_{\mathbb{R}}g\left(z,X(T),Y(T),\Phi(T,z),\bar{b}\right)\mathrm{d}z\Big\}\mathrm{d}\mathbb{P}_{\bar{B}}\Big]\end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}}\Big\{\frac{\partial f\left(t,z,\bar{b}\right)}{\partial y}\Gamma_1(t)+\frac{\partial f\left(t,z,\bar{b}\right)}{\partial x}\Gamma_2(t)+\frac{\partial f\left(t,z,\bar{b}\right)}{\partial \phi}\Gamma_3(t,z)\Big\}\mathrm{d}z\mathrm{d}t\mathrm{d}\mathbb{P}_{\bar{B}}\Big]\\
+&\mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}}\frac{\partial f\left(t,z,\bar{b}\right)}{\partial u}\beta(t,z)\mathrm{d}z\mathrm{d}t\mathrm{d}\mathbb{P}_{\bar{B}}\Big]\\
+&\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}}\Big\{\frac{\partial g\left(z,\bar{b}\right)}{\partial y}\Gamma_1(T)+\frac{\partial g\left(z,\bar{b}\right)}{\partial x}\Gamma_2(T)+\frac{\partial g\left(z,\bar{b}\right)}{\partial \phi}\Gamma_3(T,z)\Big\}\mathrm{d}z\mathrm{d}\mathbb{P}_{\bar{B}}\Big]\\
=&I_1+I_2+I_3.
\end{aligned}
\tag{3.22}
$$

Using the notation in the preceding section, we have

$$
\begin{aligned}
I_1=&\mathbb{E}\Big[\int_{\mathbb{R}}\int_0^T\Big\{\Gamma_1(t)\Big(\frac{\partial H(t,z)}{\partial y}-p_1(t,z)\frac{\partial b_1(t)}{\partial y}-q_1(t,z)\frac{\partial \sigma_1(t)}{\partial y}\Big)\\
&+\Gamma_2(t)\Big(\frac{\partial H(t,z)}{\partial x}-p_2(t,z)\frac{\partial b_2(t)}{\partial x}-q_2(t,z)\frac{\partial \sigma_2(t)}{\partial y}\Big)\\
&+\Gamma_3(t,z)\Big(\frac{\partial H(t,z)}{\partial \phi}-p_3(t,z)\frac{\partial b_3(t,z)}{\partial \phi}-q_3(t,z)\frac{\partial \sigma_3(t,z)}{\partial \phi}\Big)\Big\}\mathrm{d}t\mathrm{d}z\Big].
\end{aligned}
\tag{3.23}
$$

On the other hand, using Itô's formula, we have

$$
\begin{aligned}
I_3=&\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}}\Big\{\frac{\partial g\left(z,\bar{b}\right)}{\partial y}\Gamma_1(T)+\frac{\partial g\left(z,\bar{b}\right)}{\partial x}\Gamma_2(T)+\frac{\partial g\left(z,\bar{b}\right)}{\partial \phi}\Gamma_3(T,z)\Big\}\mathrm{d}z\mathrm{d}\mathbb{P}_{\bar{B}}\Big]\\
=&\mathbb{E}\Big[\int_{\mathbb{R}}p_1(T,z)\Gamma_1(T)+p_2(T,z)\Gamma_2(T)+p_3(T,z)\Gamma_3(T,z)\mathrm{d}z\Big]\\
=&\mathbb{E}\Big[\int_{\mathbb{R}}\Big(\int_0^T\Big\{-\frac{\partial H(t,z)}{\partial y}\Gamma_1(t)+p_1(t,z)\Gamma_1(t)\frac{\partial b_1(t)}{\partial y}+p_1(t,z)\beta(t,z)\frac{\partial b_1(t)}{\partial u}\\
&+q_1(t,z)\Big(\frac{\partial \sigma_1(t)}{\partial y}\Gamma_1(t)+\frac{\partial \sigma_1(t)}{\partial u}\beta(t,z)\Big)\Big\}\mathrm{d}t\\
&+\int_0^T\Big\{-\frac{\partial H(t,z)}{\partial x}\Gamma_2(t)+p_2(t,z)\Gamma_2(t)\frac{\partial b_2(t)}{\partial x}+p_2(t,z)\beta(t,z)\frac{\partial b_2(t)}{\partial u}\\
&+q_2(t,z)\Big(\frac{\partial \sigma_2(t)}{\partial x}\Gamma_2(t)+\frac{\partial \sigma_2(t)}{\partial u}\beta(t,z)\Big)\Big\}\mathrm{d}t\\
&+\int_0^T\Big\{-\Big(L^*p_3(t,z)+\frac{\partial H(t,z)}{\partial \phi}-\frac{\partial}{\partial z}\Big(\frac{\partial H(t,z)}{\partial \phi'}\Big)\Big)\Gamma_3(t,z)\\
&+p_3(t,z)\Big(L\Gamma_3(t,z)+\Gamma_3(t,z)\frac{\partial b_3(t,z)}{\partial \phi}+\frac{\partial b_3(t,z)}{\partial \phi'}\frac{\partial \Gamma_3(t,z)}{\partial z}+\beta(t,z)\frac{\partial b_3(t,z)}{\partial u}\Big)\\
&+q_3(t,z)\Big(\Gamma_3(t,z)\frac{\partial \sigma_3(t,z)}{\partial \phi}+\frac{\partial \sigma_3(t,z)}{\partial \phi'}\frac{\partial \Gamma_3(t,z)}{\partial z}+\beta(t,z)\frac{\partial \sigma_3(t,z)}{\partial u}\Big)\Big\}\mathrm{d}t\Big)\mathrm{d}z\Big].
\end{aligned}
\tag{3.24}
$$

Combining (3.24) and (3.23) yields

$$
\begin{aligned}
&I_1+I_2+I_3\\
=&\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}\Big\{-\Big(\Gamma_3(t,z)L^*p_3(t,z)-\Gamma_3(t,z)\frac{\partial}{\partial z}\Big(\frac{\partial H(t,z)}{\partial \phi'}\Big)\Big)\\
&+p_3(t,z)L\Gamma_3(t,z)+\frac{\partial \Gamma_3(t,z)}{\partial z}\frac{\partial b_3(t,z)}{\partial \phi'}p_3(t,z)+\frac{\partial \Gamma_3(t,z)}{\partial z}\frac{\partial \sigma_3(t,z)}{\partial \phi'}q_3(t,z)\\
&+\Big(p_1(t,z)\frac{\partial b_1(t)}{\partial u}+q_1(t,z)\frac{\partial \sigma_1(t)}{\partial u}+p_2(t,z)\frac{\partial b_2(t)}{\partial u}+q_2(t,z)\frac{\partial \sigma_2(t)}{\partial u}\\
&+q_3(t,z)\frac{\partial b_3(t,z)}{\partial u}+q_3(t,z)\frac{\partial \sigma_3(t,z)}{\partial u}+\frac{\partial f(t,z)}{\partial u}\Big)\beta(t,z)\Big\}\mathrm{d}z\mathrm{d}t\Big]\\
=&\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}\Big\{-\Big(p_3(t,z)L\Gamma_3(t,z)+\frac{\partial \Gamma_3}{\partial z}(t,z)\Big(\frac{\partial b_3(t,z)}{\partial \phi'}p_3(t,z)+\frac{\partial \sigma_3(t,z)}{\partial \phi'}q_3(t,z)\Big)\Big)\\
&+p_3(t,z)L\Gamma_3(t,z)+\frac{\partial \Gamma_3(t,z)}{\partial z}\Big(\frac{\partial b_3(t,z)}{\partial \phi'}p_3(t,z)+\frac{\partial \sigma_3(t,z)}{\partial \phi'}q_3(t,z)\Big)\\
&+\frac{\partial H(t,z)}{\partial u}\beta(t,z)\Big\}\mathrm{d}z\mathrm{d}t\Big]\\
=&\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}\beta(t,z)\frac{\partial H(t,z)}{\partial u}\mathrm{d}z\mathrm{d}t\Big].
\end{aligned}
\tag{3.25}
$$

This holds in particular for $\beta(t,z,\omega)\in\mathcal{U}_{ad}$ of the form

$$\beta(t,z,\omega)=\alpha(\omega)\chi_{[s,T]}(t)\zeta(z);\ t\in[0,T]$$

for a fixed $s\in[0,T)$, where α is a bounded $\mathcal{F}_s$-measurable random variable and $\zeta(z)\in\mathbb{R}$ is bounded and deterministic. This gives

$$\mathbb{E}\Big[\int_{\mathbb{R}}\int_s^T\frac{\partial H(t,z)}{\partial u}\zeta(z)\mathrm{d}t\mathrm{d}z\times\alpha\Big]=0. \tag{3.26}$$

Differentiating with respect to s, we get

$$\mathbb{E}\Big[\int_{\mathbb{R}}\frac{\partial H(s,z)}{\partial u}\zeta(z)\mathrm{d}z\mathrm{d}\times\alpha\Big]=0. \tag{3.27}$$

Since this holds for all bounded $\mathcal{F}_s$-measurable α and all bounded deterministic ζ, we conclude that

$$\mathbb{E}\Big[\frac{\partial H(t,z)}{\partial u}\Big|\mathcal{F}_t\Big]=0 \text{ for a.a., } (t,z)\in[0,T]\times\mathbb{R}.$$

Hence

$$\frac{\partial H(t,z)}{\partial u}=0 \text{ for a.a., } (t,z)\in[0,T]\times\mathbb{R},$$

since all the coefficients in $H(t,z)$ are $\mathcal{F}_t$-adapted. It follows that (i)$\Rightarrow$(ii).

(ii) $\Rightarrow$ (i). Assume that there exists $u\in\mathcal{U}_{ad}$ such that (3.21) holds. By reversing the argument, we have that (3.27) holds and hence (3.26) is also true. Hence, we have that (3.25) holds for all $\beta(t,z,\omega)=\alpha(\omega)\chi_{[s,T]}(t)\zeta(z)\in\mathcal{U}_{ad}$ that is

$$\mathbb{E}\Big[\int_{\mathbb{R}}\int_s^T\frac{\partial H(t,z)}{\partial u}\zeta(z)\mathrm{d}t\mathrm{d}z\times\alpha\Big]=0$$

for some $s \in [0,T]$, some bounded $\mathcal{F}_s$-measurable random variable α and some bounded and deterministic $\zeta(z) \in \mathbb{R}$. Hence the above equality holds for all linear combinations of such β. Using the fact that all bounded $\beta \in \mathcal{U}_{ad}$ can be approximated pointwisely in (t,z,ω) by such linear combination, we obtain that (3.25) holds for all bounded $\beta \in \mathcal{U}_{ad}$. Therefore, by reversing the previous arguments in the remaining part of the proof, we get that

$$\frac{\mathrm{d}}{\mathrm{d}s} J(u+s\beta)\Big|_{s=0} = 0 \text{ for all bounded } \ \beta \in \mathcal{U}_{ad}$$

and therefore **(ii)** $\Rightarrow$ **(i)**. □

Remark 3.5. *Example of systems not satisfying concavity assumption are regime switching systems; see for example* [16, 18].

4. Application to hedging and pricing factor model for commodity

In this section, we apply the results and ideas developed in the previous sections to solve optimal investment problem and pricing for convenience yield model with partial observations. The model is that of Section 2.

We consider the following partial observation market:

$$\text{(Riskless asset) } \mathrm{d}S^0(t) = S^0(t)r(t)\mathrm{d}t, \tag{4.1}$$

$$\text{(observed spot price) } \mathrm{d}\tilde{S}(t) = \left(r(t) - Z(t)\right)\tilde{S}(t)\mathrm{d}t + \sigma\tilde{S}(t)\mathrm{d}W^1(t), \tag{4.2}$$

$$\text{(unobserved yield) } \mathrm{d}Z(t) = k\left(\theta - Z(t)\right)\mathrm{d}t + \rho\gamma\mathrm{d}W^1(t) + \sqrt{1-\rho^2}\gamma\mathrm{d}W^{\perp}(t). \tag{4.3}$$

where $W^{\perp}(t) = \{W^{\perp}(t), t \in [0,T]\}$ is a standard Brownian motion on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ independent of $W^1(t)$ and $r(t) = \{r(t), t \in [0,T]\}$ is the short rate assumed to be deterministic. Let $u(t)$ be a portfolio representing the amount of wealth invested in the risky asset at time t. Then the dynamics of the wealth process is given by

$$\mathrm{d}X(t) = \left(r(t)X(t) - Z(t)u(t)\right)\mathrm{d}t + \sigma u(t)\mathrm{d}W^1(t), \ \ X(0) = x. \tag{4.4}$$

A portfolio u is admissible if $u \in \mathcal{U}_{ad}$ as described in (2.11). The problem of the investor is to find $\hat{u} \in \mathcal{U}_{ad}$ such that

$$\sup_{u\in\mathcal{U}_{ad}} \mathbb{E}\Big[U\Big(X^{x,u}(T)\Big)\Big] = \mathbb{E}\Big[U\Big(X^{x,\hat{u}}(T)\Big)\Big] \tag{4.5}$$

and

$$\sup_{u\in\mathcal{U}_{ad}} \mathbb{E}\Big[U\Big(X^{x-p,u}(T) + \Pi\Big(\tilde{S}(T), B\Big)\Big)\Big] = \mathbb{E}\Big[U\Big(X^{x-p,\hat{u}}(T) + \Pi\Big(\tilde{S}(T), B\Big)\Big)\Big], \tag{4.6}$$

where $U(x) = -e^{-\lambda x}$ is the exponential utility, Π is the contingent claim on the commodity price and B is the basis risk. (4.5) (respectively (4.6)) represents the performance functional without contingent claim (respectively with claim).

We know from Section 2 that the partial observation control problem for SDE (4.1)-(4.6) can be transformed in a full observation control problem for SPDE. In this situation, we replace the process $Z(t)$ by its *unnormalized conditional density* $\Phi(t,z)$ given $\mathcal{F}_t^Y$. Then again from Section 2 the equations for the dynamics of $X, \tilde{S}$ and Φ are given by

$$\begin{aligned} \mathrm{d}X(t) =& \Big(r(t)X(t) - (r(t) - \frac{1}{2}\sigma^2)u(t)\Big)\mathrm{d}t + u(t)\sigma \mathrm{d}W(t), && (4.7) \\ \mathrm{d}\tilde{S}(t) =& \tilde{S}(t)\Big(\frac{1}{2}\sigma^2 \mathrm{d}t + \sigma \mathrm{d}W(t)\Big), && (4.8) \\ \mathrm{d}\Phi(t,z) =& \Big\{\frac{1}{2}\gamma^2\Phi''(t,z) - k\Phi(t,z) + k(\theta - z)\Phi'(t,z)\Big\}\mathrm{d}t \\ & + \Big\{r(t) - \frac{\sigma^2}{2} - z - \rho\gamma\Phi'(t,z)\Big\}\mathrm{d}W(t) \\ =& L^*\Phi(t,z)\mathrm{d}t + M^*\Phi(t,z)\mathrm{d}W(t), && (4.9) \end{aligned}$$

where $'$ represent the derivative with respect to z and W is a Brownian motion.

Recall that the objective of the investor is: find $\hat{u} \in \mathcal{U}_{ad}$ such that

$$J(\hat{u}) = \sup_{u \in \mathcal{U}_{ad}} J(u), \tag{4.10}$$

with

$$J(u) = \tilde{\mathbb{E}}\left[\int_{\mathbb{R}} U\left(X^{x,u}(T)\right)\Phi(T,z)\mathrm{d}z\right], \text{ or} \tag{4.11}$$

$$J(u) = \tilde{\mathbb{E}}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}} U\left(X^{x-p^b,u}(T) + \Pi\left(\tilde{S}(T), B(z) + \bar{b}\right)\right)\Phi(T,z)\mathrm{d}z\mathrm{d}\mathbb{P}_{\bar{B}}\Big]. \tag{4.12}$$

In the sequel, the performance functional (4.12) will be used in solving the optimisation problem (4.10) and the solution to the utility maximisation without claim will follow by setting $\Pi = 0 = p^b$. Let us observe that in the controlled state system (4.7)-(4.9), only the process X depends on the control u. In addition, the coefficients satisfy condition of existence and uniqueness of strong solutions of system (4.7)-(4.9). We wish to apply Theorem 3.3 to solve the above control problem.

We start by writing down the Hamiltonian

$$\begin{aligned} & H(t,z,x,\tilde{s},\bar{b},u,\phi,\phi',p_1,q_1,p_2,q_2,p_3,q_3) \\ =& \frac{1}{2}\sigma^2\tilde{s}p_1 + \sigma\tilde{s}q_1 + \left(rx - (r - \frac{1}{2}\sigma^2)u\right)p_2 + \sigma u q_2 \\ & + \left(-k\phi + k(\theta - z)\phi'\right)p_3 + \left(r - \frac{1}{2}\gamma^2 - z - \rho\gamma\phi'\right)q_3, \end{aligned} \tag{4.13}$$

where the adjoint processes $(p_1(t,z), q_1(t,z))$, $(p_2(t,z), q_2(t,z))$ and $(p_3(t,z), q_3(t,z))$ are given by

$$\left\{\begin{array}{lcl} \mathrm{d}p_1(t,z) &=& -\Big(\frac{1}{2}\sigma^2 p_1(t,z) + \sigma q_1(t,z)\Big)\mathrm{d}t + q_1(t,z)\mathrm{d}W(t) \\ p_1(T,z) &=& \int_{\mathbb{R}} \lambda\frac{\partial\Pi}{\partial S}\Big(\tilde{S}(T), B(z) + \bar{b}\Big)e^{-\lambda\left(X(T) + \Pi\left(\tilde{S}(T), B(z) + \bar{b}\right)\right)}\Phi(T,z)\mathrm{d}\mathbb{P}_{\bar{B}}, \end{array}\right. \tag{4.14}$$

$$\begin{cases} \mathrm{d}p_2(t,z) = -rp_2(t,z)\mathrm{d}t + q_2(t,z)\mathrm{d}W(t) \\ p_2(T,z) = \lambda\int_{\mathbb{R}} e^{-\lambda\left(X(T) + \Pi\left(\tilde{S}(T), B(z) + \bar{b}\right)\right)}\Phi(T,z)\mathrm{d}\mathbb{P}_{\bar{B}}, \end{cases} \tag{4.15}$$

and

$$\begin{cases} \mathrm{d}p_3(t,z) = -\dfrac{1}{2}\gamma^2\dfrac{\partial^2 p_3(t,z)}{\partial z^2}\mathrm{d}t + q_3(t,z)\mathrm{d}W(t) \\ p_3(T,z) = \displaystyle\int_{\mathbb{R}} e^{-\lambda\left(X(T)+\Pi\left(\tilde{S}(T),B(z)+\bar{b}\right)\right)}\mathrm{d}\mathbb{P}_{\bar{B}}. \end{cases} \tag{4.16}$$

The generators of the BSDEs (4.14) and (4.15) are linear in their arguments and thanks to [8, Proposition 2.2], the final condition belongs to a Sobolev space. Hence, there exists a unique strong solution to the BSDE (4.14) (respectively (4.15)) in an appropriate Banach space. Furthermore, the BSPDE (4.16) is classical and thus has a unique strong solution; see for example [22].

Let $\hat{u}$ be candidate for an optimal control and let $\hat{X}, \hat{\tilde{S}}, \hat{\Phi}$ be the associated optimal processes with corresponding solution $\hat{p}(t,z) = (\hat{p}_1(t,z), \hat{p}_2(t,z), \hat{p}_3(t,z))$, $\hat{q}(t,z) = (\hat{q}_1(t,z), \hat{q}_2(t,z), \hat{q}_3(t,z))$ of the adjoint equations.

Since U and Π are concave and H is linear in its arguments, it follows that the first and second conditions of Theorem 3.3 are satisfied. In the following, we use the first order condition of optimality to find an optimal control.

Using the first order condition of optimality, we have

$$\Big(r - \frac{1}{2}\sigma^2\Big)\hat{p}_2(t,z) = \sigma\hat{q}_2(t,z). \tag{4.17}$$

Since the BSDE satisfied by $(\hat{p},\hat{q}) = (p_2,q_2)$ is linear, we try a solution of the form

$$\hat{p}_2(t,z) = -e^{-\lambda\left(\hat{X}(t)e^{\int_t^T r(s)\mathrm{d}s} + \Psi(t,\hat{\tilde{S}}(t),\Phi(t,z)))\right)}, \tag{4.18}$$

where Ψ is a smooth function. For simplicity, we write $\hat{\tilde{S}} = S$. Let $\tilde{X}(t) = e^{-\lambda\hat{X}(t)e^{\int_t^T r(s)\mathrm{d}s}}$. Then using Itô's formula, we have

$$\begin{aligned} &\mathrm{d}\tilde{X}(t) \\ =& -\lambda e^{-\lambda\hat{X}(t)e^{\int_t^T r(s)\mathrm{d}s}}\mathrm{d}\Big(\hat{X}(t)e^{\int_t^T r(s)\mathrm{d}s}\Big) + \frac{1}{2}\lambda^2 e^{-\lambda\hat{X}(t)e^{\int_t^T r(s)\mathrm{d}s}}\mathrm{d}\langle\hat{X}(\cdot)e^{\int_\cdot^T r(s)\mathrm{d}s}\rangle_t \\ =& -\lambda e^{\int_t^T r(s)\mathrm{d}s}e^{-\lambda\hat{X}(t)e^{\int_t^T r(s)\mathrm{d}s}}\Big\{(\frac{\sigma^2}{2} - r(t))u(t)\mathrm{d}t + u(t)\sigma\mathrm{d}W(t) - \frac{1}{2}\lambda e^{\int_t^T r(s)\mathrm{d}s}u^2(t)\sigma^2\mathrm{d}t\Big\} \\ =& -\lambda e^{\int_t^T r(s)\mathrm{d}s}\tilde{X}(t)\Big\{\Big((\frac{\sigma^2}{2} - r(t))u(t) - \frac{1}{2}\lambda e^{\int_t^T r(s)\mathrm{d}s}u^2(t)\sigma^2\Big)\mathrm{d}t + u(t)\sigma\mathrm{d}W(t)\Big\}. \end{aligned} \tag{4.19}$$

On the other hand, applying the Itô's formula to the two dimensional process (S,Φ), we have

$$
\begin{aligned}
&\mathrm{d}\Big(e^{-\lambda\Psi(t,S(t),\Phi(t,z))}\Big)\\
=&-\lambda e^{-\lambda\Psi(t,S(t),\Phi(t,z))}\mathrm{d}\Psi(t,S(t),\Phi(t,z))+\frac{1}{2}\lambda^2 e^{-\lambda\Psi(t,S(t),\Phi(t,z))}\mathrm{d}\langle\Psi(t,S(t),\Phi(t,z))\rangle_t\\
=&-\lambda e^{-\lambda\Psi(t,S(t),\Phi(t,z))}\Big\{\Psi_t(t,S(t),\Phi(t,z))\mathrm{d}t+\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\Big(\frac{1}{2}\sigma^2\mathrm{d}t+\sigma\mathrm{d}W(t)\Big)\\
&+\frac{1}{2}\frac{\partial^2\Psi}{\partial S^2}(t,S(t),\Phi(t,z))S^2(t)\sigma^2\mathrm{d}t+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))L^*\Phi(t,z)\mathrm{d}t\\
&+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\mathrm{d}W(t)+\frac{1}{2}\frac{\partial^2\Psi}{\partial\Phi^2}(t,S(t),\Phi(t,z))(M^*\Phi(t,z))^2\mathrm{d}t\\
&+\frac{\partial^2\Psi}{\partial\Phi\partial S}(t,S(t),\Phi(t,z))\sigma S(t)M^*\Phi(t,z)\mathrm{d}t\Big\}\\
&+\frac{1}{2}\lambda^2 e^{-\lambda\Psi(t,S(t),\Phi(t,z))}\Big\{\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big\}^2\mathrm{d}t\\
=&-\lambda e^{-\lambda\Psi(t,S(t),\Phi(t,z))}\Big(\Big\{\Psi_t(t,S(t),\phi(t,z))+\frac{1}{2}\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma^2\\
&+\frac{1}{2}\frac{\partial^2\Psi}{\partial S^2}(t,S(t),\Phi(t,z))S^2(t)\sigma^2+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))L^*\Phi(t,z)\\
&+\frac{1}{2}\frac{\partial^2\Psi}{\partial\Phi^2}(t,S(t),\Phi(t,z))(M^*\Phi(t,z))^2+\frac{\partial^2\Psi}{\partial\Phi\partial S}(t,S(t),\Phi(t,z))\sigma S(t)M^*\Phi(t,z)\\
&-\frac{1}{2}\lambda\Big(\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big)^2\Big\}\mathrm{d}t\\
&+\Big\{\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big\}\mathrm{d}W(t)\Big).
\end{aligned}
\tag{4.20}
$$

Combining (4.19) and (4.20) and using product rule, we have

$$
\begin{aligned}
&\mathrm{d}p_2(t,z)\\
=&\tilde{X}(t)\lambda e^{-\lambda\Psi(t,S,\Phi)}\Big(\Big\{\Psi_t(t,S(t),\phi(t,z))+\frac{1}{2}\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma^2\\
&+\frac{1}{2}\frac{\partial^2\Psi}{\partial S^2}(t,S(t),\Phi(t,z))S^2(t)\sigma^2+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))L^*\Phi(t,z)\\
&+\frac{1}{2}\frac{\partial^2\Psi}{\partial\Phi^2}(t,S(t),\Phi(t,z))(M^*\Phi(t,z))^2+\frac{\partial^2\Psi}{\partial\Phi\partial S}(t,S(t),\Phi(t,z))\sigma S(t)M^*\Phi(t,z)\\
&-\frac{1}{2}\lambda\Big(\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big)^2\Big\}\mathrm{d}t\\
&+\Big\{\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big\}\mathrm{d}W(t)\Big) \qquad (4.21)\\
&+e^{-\lambda\Psi(t,S(t),\Phi(t,z))}\lambda e^{\int_t^T r(s)\mathrm{d}s}\tilde{X}(t)\Big\{\Big((\frac{1}{2}\sigma^2-r(t))u(t)-\frac{1}{2}\lambda e^{\int_t^T r(s)\mathrm{d}s}u^2(t)\sigma^2\Big)\mathrm{d}t+u(t)\sigma\mathrm{d}W(t)\Big\}\\
&-\lambda^2 e^{\int_t^T r(s)\mathrm{d}s}\tilde{X}(t)e^{-\lambda\Psi(t,S(t),\Phi(t,z))}u(t)\sigma\Big\{\frac{\partial\Psi}{\partial S}(t,S(t),\Phi(t,z))S\sigma+\frac{\partial\Psi}{\partial\Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big\}\mathrm{d}t.
\end{aligned}
$$

From this, we get

$$\begin{aligned}
\mathrm{d}p_2(t,z) =& -\lambda p_2(t,z)\Big[\Big\{\Psi_t(t,S(t),\phi(t,z)) + \frac{1}{2}\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma^2 \\
&+ \frac{1}{2}\frac{\partial^2 \Psi}{\partial S^2}(t,S(t),\Phi(t,z))S^2(t)\sigma^2 + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))L^*\Phi(t,z) \\
&+ \frac{1}{2}\frac{\partial^2 \Psi}{\partial \Phi^2}(t,S(t),\Phi(t,z))(M^*\Phi(t,z))^2 + \frac{\partial^2 \Psi}{\partial \Phi \partial S}(t,S(t),\Phi(t,z))\sigma S(t) M^*\Phi(t,z) \\
&- \frac{1}{2}\lambda\Big(\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big)^2 \\
&- e^{\int_t^T r(s)\mathrm{d}s}\lambda u(t)\sigma\Big(\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S\sigma + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big) \\
&+ e^{\int_t^T r(s)\mathrm{d}s}\Big((\frac{1}{2}\sigma^2 - r(t))u(t) - \frac{1}{2}\lambda e^{\int_t^T r(s)\mathrm{d}s}u^2(t)\sigma^2\Big)\Big\}\mathrm{d}t \qquad (4.22) \\
&+ \Big\{\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z) + e^{\int_t^T r(s)\mathrm{d}s}u(t)\sigma\Big\}\mathrm{d}W(t)\Big].
\end{aligned}$$

Comparing (4.22) and (4.15), we get that Ψ must satisfy the following differential equation:

$$\begin{aligned}
r =& \lambda\Big\{\Psi_t(t,S(t),\phi(t,z)) + \frac{1}{2}\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma^2 \\
&+ \frac{1}{2}\frac{\partial^2 \Psi}{\partial S^2}(t,S(t),\Phi(t,z))S^2(t)\sigma^2 + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))L^*\Phi(t,z) \\
&+ \frac{1}{2}\frac{\partial^2 \Psi}{\partial \Phi^2}(t,S(t),\Phi(t,z))(M^*\Phi(t,z))^2 + \frac{\partial^2 \Psi}{\partial \Phi \partial S}(t,S(t),\Phi(t,z))\sigma S(t) M^*\Phi(t,z) \\
&- \frac{1}{2}\lambda\Big(\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big)^2 \\
&- e^{\int_t^T r(s)\mathrm{d}s}\lambda u(t)\sigma\Big(\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S\sigma + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big) \\
&+ e^{\int_t^T r(s)\mathrm{d}s}\Big((\frac{1}{2}\sigma^2 - r(t))u(t) - \frac{1}{2}\lambda e^{\int_t^T r(s)\mathrm{d}s}u^2(t)\sigma^2\Big)\Big\}, \qquad (4.23)
\end{aligned}$$

with

$$\Psi(T,S,\Phi) = -\frac{1}{\lambda}\ln\Big(\lambda\int_{\mathbb{R}} e^{-\lambda\Pi\left(\tilde{S}(T),B(z)+\bar{b}\right)}\Phi(T,z)\mathrm{d}\mathbb{P}_{\bar{B}}\Big)$$

and

$$q_2(t,z) = -p_2(t,z)\Big\{\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z) + e^{\int_t^T r(s)\mathrm{d}s}u(t)\sigma\Big\}. \qquad (4.24)$$

Substituting (4.24) into (4.17), we get

$$(r(t) - \frac{1}{2}\sigma^2) = -\sigma\Big\{\frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S(t)\sigma + \frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z) + e^{\int_t^T r(s)\mathrm{d}s}u(t)\sigma\Big\},$$

i.e.,

$$\begin{aligned}
\hat{u}(t) =& \hat{u}(t,z) \qquad (4.25) \\
=& e^{-\int_t^T r(s)\mathrm{d}s}\Big\{\frac{1}{\sigma^2}\Big(r(t) - \frac{\sigma^2}{2}\Big) + \frac{\partial \Psi}{\partial S}(t,S(t),\Phi(t,z))S(t) + \frac{1}{\sigma}\frac{\partial \Psi}{\partial \Phi}(t,S(t),\Phi(t,z))M^*\Phi(t,z)\Big\}.
\end{aligned}$$

Hence the total value invested is the cost invested in the risky asset and another cost due to partial observation.

Remark 4.1. *Assume that there is no claim, then*

$$\hat{u}_0(t) = \hat{u}_0(t,z) = e^{-\int_t^T r(s)\mathrm{d}s}\Big\{\frac{1}{\sigma^2}\Big(r(t) - \frac{\sigma^2}{2}\Big) + \frac{1}{\sigma}\frac{\partial\Psi}{\partial\Phi}(t,\Phi(t,z))M^*\Phi(t,z)\Big\}. \tag{4.26}$$

We have shown the following :

Theorem 4.2. *The optimal portfolio $\hat{u} \in \mathcal{A}_{ad}$, to the partial observation utility maximisation control problem* (2.1)-(2.9) (*respectively* (2.1)-(2.8)) *is given by* (4.25) (*respectively* (4.26)).

Assume that the interest rate is constant. The terminal wealth with initial value x can be expressed as

$$X^x(T) = xe^{rT} - \int_0^T e^{r(T-t)}(r - \frac{1}{2}\sigma^2)u(t)\mathrm{d}t + \int_0^T e^{r(T-t)}u(t)\sigma\mathrm{d}W(t) \tag{4.27}$$

and the wealth with initial value $x - p^b$ is given by

$$X^{x-p^b}(T) = xe^{rT} - p^b e^{rT} - \int_0^T e^{r(T-t)}(r - \frac{1}{2}\sigma^2)u(t)\mathrm{d}t + \int_0^T e^{r(T-t)}u(t)\sigma\mathrm{d}W(t). \tag{4.28}$$

Since the wealth process is the only process depending on the control in the utility maximisation problems (2.8)-(2.9), we have the following result for the utility indifference price.

Theorem 4.3. *Assume that the interest rate is constant. The price indifference p^b for the buyer of the claim $\Pi = \Pi\Big(\tilde{S}(t), B(z) + \bar{b}\Big)$ is given by*

$$p^b = -\frac{e^{-rT}}{\lambda}\ln\left(\frac{\tilde{\mathbb{E}}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}}\exp\lambda\Big(\int_0^T e^{r(T-t)}(r-\frac{1}{2}\sigma^2)\hat{u}_0(t)\mathrm{d}t - \int_0^T e^{r(T-t)}\hat{u}_0(t)\sigma\mathrm{d}W(t)\Big)\Phi(T,z)\mathrm{d}z\mathrm{d}\mathbb{P}_{\bar{B}}\Big]}{\tilde{\mathbb{E}}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}}\exp\lambda\Big(\int_0^T e^{r(T-t)}(r-\frac{1}{2}\sigma^2)\hat{u}(t)\mathrm{d}t - \int_0^T e^{r(T-t)}\hat{u}(t)\sigma\mathrm{d}W(t)\Big)e^{-\lambda\Pi}\Phi(T,z)\mathrm{d}z\mathrm{d}\mathbb{P}_{\bar{B}}\Big]}\right), \tag{4.29}$$

where $\hat{u}$ and $\hat{u}_0$ are given by (4.25) *and* (4.26) *respectively.*

5. Conclusion

In this paper, we have derived a sufficient and equivalent stochastic maximum principle for an optimal control problem for partially observed systems. The existence of correlated noise between the control and the observations systems lead to a degenerated Zakai equation and hence the need of results on existence of unique strong solutions of such equations. Based on the existence results, we are able to give a sufficient and equivalent "strong" maximum principle. The results obtained are then applied to study a hedging and pricing problem for partially observed convenience yield model. The coefficients of the controlled and observation processes studied in this paper are time independent and it will be of great interest to consider time dependent coefficients due to seasonality factors. Furthermore, dependence of jumps of the commodity price has recently been studied, hence extension to systems with jumps is necessary and will be the object of future research. Using a more general system could also lead to optimal control depending on adjoint equations and hence the need of numerical implementation of BSPDE with jumps to find values of the optimal portfolio and utility indifference price when the parameters are known.

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